GRADIENT MAP OF ISOPARAMETRIC POLYNOMIAL AND ITS APPLICATION TO GINZBURG-LANDAU SYSTEM

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Abstract. In this note, we study properties of the gradient map of the isoparametric polynomial. For a given isoparametric hypersurface in sphere, we calculate explicitly the gradient map of its isoparametric polynomial which turns out many interesting phenomenons and applications. We find that it should map not only the focal submanifolds to focal submanifolds, isoparametric hypersurfaces to isoparametric hypersurfaces, but also map isoparametric hypersurfaces to focal submanifolds. In particular, it turns out to be a homogeneous polynomial automorphism on certain isoparametric hypersurface. As an immediate consequence, we get the Brouwer degree of the gradient map which was firstly obtained by Peng and Tang with moving frame method. Following Farina's construction, another immediate consequence is a counter example of the Brézis question about the symmetry for the Ginzburg-Landau system in dimension 6, which gives a partial answer toward the Open problem 2 raised by Farina.

1. Introduction

Let M be a connected oriented isoparametric hypersurface in the unit sphere S^{n+1} with g distinct principle curvatures. The isoparametric polynomial F of M is a homogeneous polynomial of degree g in the Euclidean space $\mathbb{R}^{n+2}$, which is uniquely determined by M and satisfies the Cartan-Münzner equations:

$$|\nabla F|^2 = g^2|x|^{2g-2}, \tag{1}$$

$$\Delta F = \frac{g^2}{2}(m_2 - m_1)|x|^{g-2}, \tag{2}$$

where ∇F, ΔF denote the gradient and Laplacian of F in $\mathbb{R}^{n+2}$ respectively, and m_1, m_2 the multiplicities of the maximal and minimal principal curvature of M.

Cartan (see [Car38], [Car39]) considered isoparametric hypersurfaces in spheres and solved the classification problem in the case $g \in \{1, 2, 3\}$. By using delicate cohomological and algebraic arguments, Münzner (see [Mün80], [Mün81]) obtained the

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splendid result that the number g must be $1, 2, 3, 4$ or 6. Deep going study about the geometry and topology of isoparametric hypersurfaces leads to a lot of important results (see [TT72], [Tak76], [OT76], [Abr83], [DN85], [Tan91], [Miy93], [CCJ07], [Imm08], *etc.*).

It is well known that there is a one-to-one correspondence between isoparametric polynomials and families of parallel isoparametric hypersurfaces. Throughout of this paper, we identify both isoparametric polynomials and isoparametric hypersurfaces with their congruences under isometries of the Euclidean space. Thus we say an isoparametric polynomial is unique means that it's unique under congruence. Given an isoparametric hypersurface M, one can construct a function F which turns out to be an isoparametric polynomial with its level hypersurfaces being parallel hypersurfaces of M. Given an isoparametric polynomial F on $\mathbb{R}^{n+2}$, let f denote the restriction to S^{n+1}. Then the level hypersurfaces of f consist of a family of parallel isoparametric hypersurfaces in the sphere. From the Cartan-Münzner equation (1), it is not difficult to show that f must have $[-1, 1]$ as its range. Moreover, the gradient of f on S^{n+1} can vanish only when $f = \pm 1$, and for each $s \in (-1, 1)$, the level set

$$M_s = \{p \in S^{n+1} | f(p) = s\}$$

is a compact connected isoparametric hypersurface, while M_1, M_{-1} are the focal submanifolds in the sphere with codimension $m_1 + 1$ and $m_2 + 1$ respectively. In other words, the level sets of f give a "singular" foliation of S^{n+1} as $S^{n+1} = \bigcup_{s\in[-1,1]} M_s$.

The gradient map Φ is a map from $\mathbb{R}^{n+2}$ to $\mathbb{R}^{n+2}$ defined by $\Phi = \frac{1}{g}\nabla F$. Obviously, each component of Φ is a homogeneous polynomial of degree $g - 1$, and the restriction of Φ to S^{n+1} provides a homogeneous polynomial map from S^{n+1} to S^{n+1}. In [PT96], Peng and Tang applied the moving frame method successfully to obtain the Brouwer degree of Φ. In this paper, we try to study further properties of the gradient map of the isoparametric polynomial and establish

Theorem 1.1. Let F be an isoparametric polynomial of degree g on $\mathbb{R}^{n+2}$ ($g \geqslant 2$), the gradient map $\Phi = \frac{1}{g}\nabla F$. Then $\Phi(M_{\cos(g\tau)}) = M_{\cos(g(1-g)\tau)}$. In particular,

(i) Φ maps focal submanifold to focal submanifold;
(ii) if $s \in \mathcal{D} = \{\cos\frac{k\pi}{g-1} | 1 \leqslant k < g - 1\}$, Φ maps isoparametric hypersurface M_s to focal submanifold;
(iii) if $s \in (-1, 1) - \mathcal{D}$, Φ maps isoparametric hypersurface M_s to isoparametric hypersurface. In particular, Φ provides a homogeneous polynomial automorphism on certain isoparametric hypersurface.

As an immediate consequence, we can get the Brouwer degree of the gradient map.

In Section 3, we give an application of the gradient maps to the Open problem 2 of Farina [Far04] about Brézis question on the symmetry for the Ginzburg-Landau system.

Question (Brézis [Bré99]**)** Let $u : \mathbb{R}^N \to \mathbb{R}^N$ be a solution of

$$\Delta u = u(|u|^2 - 1) \quad \texttt{on } \mathbb{R}^N, \quad N \geqslant 3 \tag{3}$$

with $|u(x)| \to 1$ as $|x| \to \infty$ (possibly with a "good" rate of convergence). Assume $\deg(u, \infty) = \pm 1$. Does u have the form

$$u(x) = \frac{x}{|x|} h(|x|) \tag{4}$$

(modulo translation and isometry), where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function, such that $h(0) = 0$ and $h(\infty) = 1$?

In [Bré99], Brézis gave an affirmative answer for the case $N = 2$ and thus raised the above question. For the case $N = 8$, Farina [Far04] gave a negative answer to it. In fact, he constructed a radial solution which can be written as the form $u(x) = G(\frac{x}{|x|})h(|x|)$. Therefore, he formulated his Open problem 2 which is to study Brézis question in dimension $N \geqslant 3$ and $N \neq 8$. Following Farina's construction and using the gradient map of some isoparametric polynomial, we give another counter example.

Theorem 1.2. There exists a solution $u : \mathbb{R}^6 \to \mathbb{R}^6$, of the Ginzburg-Landau system (3), satisfying

(i) $|u(x)| \to 1$ as $|x| \to \infty$;
(ii) $\deg(u, \infty) = 1$,

which has not the form (4) (modulo translation and isometry).

Furthermore, u is a radial solution, *i.e.*, it can be written in the following way:

$$u(x) = \Phi\left(\frac{x}{|x|}\right) h(|x|), \tag{5}$$

where Φ is the gradient map of the isoparametric polynomial in $\mathbb{R}^6$ with $g = 4, m_1 = m_2 = 1$, and $h \in C^2(\mathbb{R}_+, \mathbb{R})$ is the unique solution of

$$\begin{cases} -h'' - 5\frac{h'}{r} + 21\frac{h}{r^2} = h(1 - h^2), \ r > 0, \\ h(0) = 0, \qquad h(\infty) = 1. \end{cases} \tag{6}$$

Remark 1.1. Takagi [Tak76] proved that for the isoparametric polynomial with $g = 4$, if one of the principal curvatures of M has multiplicity one, then M must be homogeneous. Hence, the isoparametric polynomial in Theorem 1.2 is unique and one can write it as follows,

$$F = (|x|^2 + |y|^2)^2 - 2\{(|x|^2 - |y|^2)^2 + 4\langle x, y\rangle^2\},$$

where $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $\langle , \rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$. Thus the map $\Phi = \frac{\nabla F}{4}$ in the theorem can be written explicitly.

Remark 1.2. In the case $g = 6, m_1 = m_2 = 1$, Dorfmeister and Neher [DN85] showed that the isoparametric hypersurface in S^7 must be homogeneous. In [Miy93], Miyaoka gave an interesting description in this case. She found that a homogeneous hypersurface in S^7 with $g = 6$ is the inverse image of an isoparametric hypersurface in S^4 with $g = 3$ under Hopf fibering. On the other hand, Cartan [Car38] [Car39] determined all isoparametric hypersurfaces with $g = 3$. In particular, when $m_1 = m_2 = 1$, the isoparametric polynomial in $\mathbb{R}^5$ can be written as

$$F(x, y, X, Y, Z) = x^3 - 3xy^2 + \frac{3}{2}x(X^2 + Y^2 - 2Z^2) + \frac{3\sqrt{3}}{2}y(X^2 - Y^2) + 3\sqrt{3}XYZ,$$

where $(x, y, X, Y, Z) \in \mathbb{R}^5$. Therefore, we can write the isoparametric polynomial with $g = 6, m_1 = m_2 = 1$ as below

$$\tilde{F} = F \circ \pi,$$

where $\pi : \mathbb{R}^8 \to \mathbb{R}^5$ is given by

$$\pi(u, v) = (|u|^2 - |v|^2, 2u\bar{v}), \quad u, v \in \text{the quaternion field } H \cong \mathbb{R}^4.$$

The gradient map $\Phi = \frac{\nabla \tilde{F}}{6} : \mathbb{R}^8 \to \mathbb{R}^8$ is exactly the map G in Farina's counter example as we mentioned before.

2. Gradient map of isoparametric polynomial

In this section, following Münzner[Mün80], firstly we'll construct the isoparametric polynomial from a given isoparametric hypersurface (See also [CR85]).

Suppose M is a connected oriented isoparametric hypersurface in S^{n+1} with g distinct principal curvatures $\lambda_i := \cot(\theta_i)$, $\lambda_1 > \cdots > \lambda_g$. It's well known that

$$\theta_i = \theta_1 + \frac{i-1}{g}\pi, \quad i = 1, \cdots, g,$$

and the multiplicity m_i of λ_i satisfy: $m_i = m_{i+2},\ m_2 = m_g$.

Let ξ be the oriented unit normal vector field of M. Consider the normal exponential map $\phi : M \times \mathbb{R} \to S^{n+1}$ defined by

$$\phi(x, t) = \cos t \cdot x + \sin t \cdot \xi_x. \tag{7}$$

We know that ϕ has rank $n + 1$, except where $\cot t$ is a principal curvature of M. For any regular point (x,t) of ϕ, there exists an open neighborhood U of (x, t), such that ϕ is a diffeomorphism of U onto $V = \phi(U)$. Define $\tau : V \to \mathbb{R}$ by

$$\tau(p) = \theta_1 - t, \tag{8}$$

where t is considered as a function on V under the map ϕ. Note that the function τ is invariant if we take a parallel isoparametric hypersurface of M instead of M in the

definition. Then we can obtain a homogeneous function F on the cone of $\mathbb{R}^{n+2}$ over V by

$$F(rp) = r^g \cos g\tau(p), \quad p \in V,\ r > 0.$$

It is well known that the function F is the restriction of a homogeneous polynomial with degree g in $\mathbb{R}^{n+2}$, which is uniquely determined by M and satisfies the Cartan-Münzner equations (1) and (2) (see [Mün80], [CR85]). This polynomial is called the isoparametric polynomial of M.

Let Φ be the (normalized) gradient map of F, *i.e.* $\Phi = \frac{\nabla F}{g}$. Since for $g = 1$, $\Phi = \nabla F$ is a constant map. Hence in the following discussion, we assume that $g = 2, 3, 4$ or 6.

Let f denote the restriction of F on S^{n+1}. For convenience, we write the level set of f as $\tilde{M}_\tau = f^{-1}(\cos g\tau)$, $\tau \in \mathbb{R}$. It is not difficult to check that,

(i) For any integer j, $\tilde{M}_0 = \tilde{M}_{\frac{2j\pi}{g}}$, $\tilde{M}_{\frac{\pi}{g}} = \tilde{M}_{\frac{(2j+1)\pi}{g}}$, and $S^{n+1} = \bigcup_{\tau \in I_j} \tilde{M}_\tau$, where $I_j = [\frac{j\pi}{g}, \frac{(j+1)\pi}{g}]$;

(ii) $\tilde{M}_0$, $\tilde{M}_{\frac{\pi}{g}}$ are the focal submanifolds in S^{n+1} with codimension m_1+1, m_2+1, respectively;

(iii) For any $\tau \in (0, \frac{\pi}{g})$, $\tilde{M}_\tau$ is an isoparametric hypersurface with the maximal principal curvature $\cot\tau$, *i.e.* θ_1 of $\tilde{M}_\tau$ equals τ.

Proof of Theorem 1.1. For convenience, let $M = f^{-1}(0) = \tilde{M}_{\frac{\pi}{2g}}$, then $\theta_1 = \frac{\pi}{2g}$ and $\tilde{M}_\tau = \phi(M \times \{\frac{\pi}{2g} - \tau\})$. It is easily seen that

$$\tilde{\xi} = -\sin(\frac{\pi}{2g} - \tau) \cdot x + \cos(\frac{\pi}{2g} - \tau) \cdot \xi \tag{9}$$

is the oriented unit normal vector field of $\tilde{M}_\tau$.

We now calculate the gradient map Φ. At each point $p = \phi(x, \frac{\pi}{2g} - \tau) \in \tilde{M}_\tau \subset S^{n+1}$, $\tau \in (0, \frac{\pi}{g})$,

$$\nabla F(p) = \nabla_S f(p) + \langle p, \nabla F \rangle\, p,$$

where ∇, ∇_S are the gradient operators in $\mathbb{R}^{n+2}$ and S^{n+1} respectively.

Since F is a homogeneous polynomial of degree g and $f = \cos g\tau$, by Euler's theorem,

$$\nabla F(p) = -g \sin g\tau \cdot \nabla_S \tau(p) + gF \cdot p. \tag{10}$$

On the other hand, by (7) and (8), we have

$$\nabla_S \tau(p) = -\tilde{\xi}, \tag{11}$$

Substituting (11) to (10) implies that for each $p=\phi(x,\frac{\pi}{2g}-\tau)$,

$$\begin{aligned}(12)\qquad \Phi(p) &= \frac{1}{g}\nabla F(p)=\cos g\tau(p)\cdot p+\sin g\tau(p)\cdot\tilde{\xi}_p.\\ &= \cos(\frac{\pi}{2g}+(g-1)\tau)\cdot x+\sin(\frac{\pi}{2g}+(g-1)\tau)\cdot\xi_x\\ &= \phi(x,\frac{\pi}{2g}+(g-1)\tau),\end{aligned}$$

which follows that

$$\Phi(\tilde{M}_\tau)=\tilde{M}_{(1-g)\tau}. \qquad (13)$$

By the continuity of the gradient map Φ, equalities (12) and (13) holds for all $\tau\in[0,\frac{\pi}{g}]$. In particular, the focal submanifold $\tilde{M}_0$ is the fix point set of Φ. If g is odd, Φ maps the other focal submanifold $\tilde{M}_{\frac{\pi}{g}}$ to $\tilde{M}_{\frac{(1-g)\pi}{g}}=\tilde{M}_0$, and if g is even, Φ maps $\tilde{M}_{\frac{\pi}{g}}$ to $\tilde{M}_{\frac{(1-g)\pi}{g}}=\tilde{M}_{\frac{\pi}{g}}$. Note that Φ is just the antipodal map when restricted to the focal submanifold $\tilde{M}_{\frac{\pi}{g}}$.

For $\tau\in(0,\frac{\pi}{g})$, it follows from the equality (13), $\Phi(\tilde{M}_\tau)$ is a focal submanifold if and only if $\cos(g(1-g)\tau)=\pm1$, *i.e*,

$$\tau=\frac{k}{g(g-1)}\pi,\quad 1\leqslant k<g-1.$$

When $\cos(g\tau)=\cos(g(1-g)\tau)$, *i.e*, $g=2$, or $\tau=\frac{2k\pi}{g^2}$, or $\tau=\frac{2k\pi}{g^2-2g}$, $k\in\mathbb{Z}$, Φ maps $\tilde{M}_\tau$ to itself. In particular, there's always an isoparametric hypersurface with $g\geqslant 2$ on which Φ provides a homogeneous polynomial automorphism. These complete the proof of Theorem 1.1. □

As a consequence of Theorem 1.1, we can deduce the Brouwer degree of the gradient map $\Phi=\frac{\nabla F}{g}|_{S^{n+1}}:S^{n+1}\to S^{n+1}$, by counting the number of inverse points, counted with multiplicity ±1 which is the sign of the tangential map of Φ according whether it preserves or reverses the orientation, of any regular value point. Our method here differs from that of [PT96] where they used the integral definition of Brouwer degree and calculated it by moving frame method (See [BT82] for different equivalent definitions of Brouwer degree).

Corollary 2.1. Let Φ be the gradient map of an isoparametric polynomial with degree g. Then the Brouwer degree of Φ is given by

(i) for g=2, $\deg\Phi=(-1)^{m_1+1}$;
(ii) for g=3, $\deg\Phi=(-1)^{m_1+1}+(-1)^{m_1+m_2+1}$;
(iii) for g=4, $\deg\Phi=(-1)^{m_1+1}+(-1)^{m_1+m_2+1}+(-1)^{m_2+1}$;
(iv) for g=6, $\deg\Phi=2\cdot(-1)^{m_1+1}+(-1)^{m_1+m_2+1}+(-1)^{m_2+1}-1$.

Proof. Denote by $J = (0, \frac{\pi}{g})$, $J_k = (\frac{k-1}{g(g-1)}\pi, \frac{k}{g(g-1)}\pi)$, $1 \leqslant k \leqslant g-1$. Let

$$\mathscr{M} := S^{n+1} - (\tilde{M}_0 \cup \tilde{M}_{\frac{\pi}{g}}) = \bigcup_{\tau \in J} \tilde{M}_\tau, \quad \mathscr{M}_k := \bigcup_{\tau \in J_k} \tilde{M}_\tau.$$

Then $\mathscr{M}$, $\mathscr{M}_k$ are open subsets of S^{n+1} and Theorem 1.1 implies that $\Phi|_{\mathscr{M}_k} : \mathscr{M}_k \to \mathscr{M}$ is a diffeomorphism for each $1 \leqslant k \leqslant g-1$. Thus every point p in $\mathscr{M}$ is a regular value point of Φ and its inverse set equals $\{p_k \in \mathscr{M}_k | \Phi(p_k) = p, k = 1, \cdots, g-1.\}$ having $g-1$ points. Therefore, to calculate the Brouwer degree of Φ, we need only specify the sign of its tangential map Φ_* at each p_k.

Assume $p_k \in \tilde{M}_{\tau_k}$ for some $\tau_k \in J_k$. Then the principal curvatures of $\tilde{M}_{\tau_k}$ are given by $\lambda_{ki} = \cot(\tau_k + \frac{i-1}{g}\pi)$ with multiplicities m_i satisfying $m_i = m_{i+2}$, $m_2 = m_g$, $i = 1, \cdots, g$. Note that m_1, m_2 are determined by the isoparametric polynomial and thus are same for each k. Suppose X is a principal tangent vector of $\tilde{M}_{\tau_k}$ with respect to λ_{ki} at p_k. It is easily seen from formula (12) that

$$\Phi_*(X) = \frac{\sin((1-g)\tau_k + \frac{i-1}{g}\pi)}{\sin(\tau_k + \frac{i-1}{g}\pi)} X, \tag{14}$$

where X in the right side is regarded as the vector at p by parallel translating X from p_k to p in $\mathbb{R}^{n+2}$. On the other hand, from formulas (9), (11) and (12), we can derive directly

$$\Phi_*(\tilde{\xi}_{p_k}) = (1-g)\tilde{\xi}_p, \tag{15}$$

where $\tilde{\xi}_{p_k}$, $\tilde{\xi}_p$ are the unit normal vectors of $\tilde{M}_{\tau_k}$, $\Phi(\tilde{M}_{\tau_k}) = \tilde{M}_{(1-g)\tau_k}$ at p_k and p respectively. Notice that the tangent space of S^{n+1} at p_k (*resp.* p) is spanned by such Xs and $\tilde{\xi}_{p_k}$ (*resp.* $\tilde{\xi}_p$), it follows immediately from (14) and (15) that the sign of the tangential map Φ_* at p_k is given by

$$\text{sign } \Phi_*|_{p_k} = \begin{cases} (-1)^{\frac{k+1}{2}m_1 + \frac{k-1}{2}m_2 + 1} & \text{for k is odd,} \\ (-1)^{\frac{k}{2}m_1 + \frac{k}{2}m_2 + 1} & \text{for k is even.} \end{cases} \tag{16}$$

Combining (16) with the following formula for the definition of Brouwer degree

$$\deg(\Phi) = \sum_{k=1}^{g-1} \text{sign } \Phi_*|_{p_k},$$

we can conclude the items of Corollary 2.1. □

When the isoparametric polynomial F is harmonic, *i.e.* $m_1 = m_2 =: m$. According to the tangential map Φ_* given in (14) (15), one can calculate directly that the tension field $B(\Phi) := \text{Trace}(\nabla_S \Phi_*) = 0$, hence $\Phi|_{S^{n+1}} : S^{n+1} \to S^{n+1}$ is a harmonic map (See also [ER93] and [PT96]). For applications in next section, we now focus on harmonic isoparametric polynomials.

For $g=2$, the harmonic isoparametric polynomial is given by $F(x,y)=|x|^2-|y|^2$, and thus $\Phi(x,y)=(x,-y)$, where $(x,y)\in\mathbb{R}^{m+1}\times\mathbb{R}^{m+1}$. For $g=3$, Cartan completely classified the isoparametric polynomials and showed that $m_1=m_2=1,2,4,or\ 8$. For g=4, Abresch [Abr83] showed that harmonic isoparametric polynomials must have $m_1=m_2=1,or\ 2$. These two cases were showed to be unique by Ozeki and Takeuchi [OT76]. See Remark 1.1 for explicit representation of the one with $m=1$. For $g=6$, Münzner (see [Mün80], [Mün81]) showed that it must have $m_1=m_2$. Furthermore, Abresch [Abr83] was able to show that the common multiplicity m must be either 1 or 2. In the case $m=1$, Dorfmeister and Neher showed that it must be homogeneous. See Remark 1.2 for explicit representation for the case of $m=1$. Recently, Miyaoka [Miy08] claimed that it is also unique for the case of $m=2$.

In conclusion, by Corollary 2.1 and discussions above, we have (Compare with [Tan07])

Corollary 2.2. Harmonic isoparametric polynomial exists only when

$$(g,m)=(1,m),(2,m),(3,1),(3,2),(3,4),(3,8),(4,1),(4,2),(6,1),(6,2),$$

and for each case (except possibly the last one), it's unique under congruence. Furthermore, its gradient map $\Phi|_{S^{n+1}}:S^{n+1}\to S^{n+1}$ is a polynomial harmonic map with the Brouwer degree $\deg\Phi$

(i) $(g,m)=(1,m),\quad \deg\Phi=0;$
(ii) $(g,m)=(2,m),\quad \deg\Phi=(-1)^{m+1};$
(iii) $(g,m)=(3,1),\quad \deg\Phi=0,\qquad (g,m)=(3,2),(3,4),(3,8),\quad \deg\Phi=-2;$
(iv) $(g,m)=(4,1),\quad \deg\Phi=1,\qquad (g,m)=(4,2),\quad \deg\Phi=-3;$
(v) $(g,m)=(6,1),\quad \deg\Phi=1,\qquad (g,m)=(6,2),\quad \deg\Phi=-5.$

3. Proof of Theorem 1.2

The proof should be translated word by word from Farina [Far04] once one knows examples of harmonic isoparametric polynomial with Brouwer degree of its gradient map being ±1. For completeness, we state it as follows.

By the results of [AF97] a function u having the form (5): $u(x)=\Phi(\frac{x}{|x|})h(|x|)$, with a non-constant $\Phi\in C^2(S^{N-1},\mathbb{R}^N)$ and a profile $h\in C^2(\mathbb{R}_+,\mathbb{R})$ is a solution of the Ginzburg-Landau system (3) if

(i) $\Phi(S^{N-1})\subset S^{N-1}$,
(ii) there exists a positive integer k such that $\Phi\in(\mathscr{SH}_{k,N})^N$ (where $\mathscr{SH}_{k,N}$ is the vector space of the spherical harmonics of degree k in $\mathbb{R}^N$),

and the profile h satisfies

$$\begin{cases} -h''-(N-1)\frac{h'}{r}+k(k+N-2)\frac{h}{r^2}=h(1-h^2), r>0,\\ h(0)=0. \end{cases} \tag{17}$$

Therefore, to obtain the desired conclusion it is enough to prove the existence of a map Φ satisfying (i) and (ii) above with $N = 6$ and $k = 3$, and a corresponding profile h satisfying (17) with $h(\infty) = 1$.

Existence of Φ: Corollary 2.2 implies that the gradient map $\Phi = \frac{\nabla F}{g}$ of a harmonic isoparametric polynomial F has Brouwer degree ± 1 if and only if $g = 2$, or $(g, m) = (4, 1)$, or $(g, m) = (6, 1)$. Obviously, such map Φ satisfies properties (i) and (ii) above. As mentioned before, when $g = 2$, $\Phi(x, y) = (x, -y)$ is congruent to the identity and so is trivial. The map Φ for $(g, m) = (6, 1)$ is given explicitly in Remark 1.2 and has been applied in Farina's counter example in dimension $N = gm + 2 = 8$. The map Φ for $(g, m) = (4, 1)$ is exactly the one we apply to construct the counter example in dimension $N = 6$. See Remark 1.1 for explicit form of this map Φ.

Existence of h: In [FG00] it is proved that there is a unique solution of the problem

$$\begin{cases} -h'' - (N-1)\frac{h'}{r} + k(k+N-2)\frac{h}{r^2} = h(1-h^2), r > 0, \\ h(0) = 0, \qquad h(\infty) = 1 \end{cases} \tag{18}$$

for every integer $N \geqslant 3$ and every positive integer k. Furthermore, the profile h is a strictly increasing function. This property implies that u satisfies the condition (i) in Theorem 1.2. On the other hand, owing to the special form of the constructed radial solution u, we have that $\deg(u, \infty)$ is equal to the Brouwer degree of the map Φ. This shows that (ii) in Theorem 1.2 is also satisfied. This completes the proof of Theorem 1.2.

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